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# WDVV-like equations in $\mathcal{N} = 2$ SUSY Yang-Mills Theory

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**Abstract**

The prepotential $F(a_i)$, defining the low-energy effective action of the $SU(N)$ $\mathcal{N} = 2$ SUSY gluodynamics, satisfies an enlarged set of the WDVV-like equations $F_i F_k^{-1} F_j = F_j F_k^{-1} F_i$ for any triple $i, j, k = 1, \ldots, N-1$, where matrix $F_i$ is equal to $(F_i)_{mn} = \frac{\partial^3 F}{\partial a_i \partial a_m \partial a_n}$. The same equations are actually true for generic topological theories. In contrast to the conventional formulation, when $k$ is restricted to $k = 0$, in the proposed system there is no distinguished "first" time-variable, and the indices can be raised with the help of any "metric" $\eta^{(k)}_{mn} = (F_k)_{mn}$, not obligatory flat. All the equations (for all $i, j, k$) are true simultaneously. This result provides a new parallel between the Seiberg-Witten theory of low-energy gauge models in $4d$ and topological theories.

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# 1 Definitions

According to [1], [2] the low-energy effective action of $\mathcal{N} = 2$ SUSY Yang-Mills model (the Seiberg-Witten effective theory) is given by

$$\int d^4xd^4\theta F(\Phi_i), \tag{1}$$

where the superfield $\Phi_i = \varphi^i + \theta\sigma_{\mu\nu}\tilde{\theta}G^i_{\mu\nu} + \ldots$.

The prepotential $F$ [2] is defined in terms of a family of Riemann surfaces, endowed with the meromorphic differential $dS$. For the gauge group $G = SU(N)$ the family is [2], [3], [4]

$$\begin{aligned} w + \frac{1}{w} &= 2P_N(\lambda), \\ P_N(\lambda) &= \lambda^N + \sum_{k=1}^{N-1} h_k\lambda^{k-1}, \end{aligned} \tag{2}$$

and

$$dS = \lambda\frac{dw}{w} \tag{3}$$

The prepotential $F(a_i)$ is implicitly defined by the set of equations:

$$\begin{aligned} \frac{\partial F}{\partial a_i} &= a_i^D, \\ a_i &= \oint_{A_i} dS, \\ a_i^D &= \oint_{B_i} dS. \end{aligned} \tag{4}$$

According to [4], this definition identifies $F(a_i)$ as logarithm of (truncated) $\tau$-function of Whitham integrable hierarchy. Existing experience with Whitham hierarchies [5] implies that $F(a_i)$ should satisfy some sort of the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations [6].

# 2 The statement

Below in this paper we demonstrate that WDVV equations for the prepotential actually look like

$$F_iF_k^{-1}F_j = F_jF_k^{-1}F_i \qquad \forall i,j,k = 1,\ldots,N-1. \tag{5}$$

Here $F_i$ denotes the matrix

$$(F_i)_{mn} = \frac{\partial^3 F}{\partial a_i\partial a_m\partial a_n}. \tag{6}$$

# 3 Comments

**3.1** Let us remind, first, that the conventional WDVV equations for topological field theory express the associativity of the algebra $\phi_i\phi_j = C^k_{ij}\phi_k$ (for symmetric in $i$ and $j$ structure constants): $(\phi_i\phi_j)\phi_k = \phi_i(\phi_j\phi_k)$, or $C_iC_j = C_jC_i$, for the matrix $(C_i)^m_n \equiv C^m_{in}$. These conditions become highly non-trivial since, in topological theory, the structure constants are expressed in terms of a single prepotential $F(t_i)$: $C^l_{ij} = (\eta^{-1}_{(0)})^{kl}F_{ijk}$, and

$F_{ijk} = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k}$, while the metric is $\eta^{(0)}_{kl} = F_{0kl}$, where $\phi_0 = I$ is the unity operator. In other words, the conventional WDVV equations can be written as

$$F_i F_0^{-1} F_j = F_j F_0^{-1} F_i. \tag{7}$$

In contrast to (5), $k$ is restricted to $k = 0$, associated with the distinguished unity operator.

On the other hand, in the Seiberg-Witten theory there does not clearly exist any distinguished index $i$: all the arguments $a_i$ of the prepotential are on equal footing. Thus, if some kind of the WDVV equations holds in this case, it should be invariant under any permutation of indices $i, j, k$ – criterium satisfied by the system (5).

Moreover, the same set of equations (5) is satisfied for generic topological theory: see s.4.1 below.

**3.2** In the general theory of Whitham hierarchies [5] the WDVV equations arise also in the form (7). Again, there exists a distinguished time-variable $t_0 = x$ – associated with the first time-variable of the original KP/KdV hierarchy. Moreover, usually – in contrast to the simplest topological models – the set of these variables for the Whitham hierarchy is infinitely large. In this context our eqs.(5) state that, for peculiar subhierarchies (in the Seiberg-Witten gluodynamics, it is the Toda-chain hierarchy, associated with a peculiar set of hyperelliptic surfaces), there exists a non-trivial *truncation* of the quasiclassical $\tau$-function, when it depends on the finite number ($N - 1 = g =$ genus of the Riemann surface) of *equivalent* arguments $a_i$, and satisfies a much wider set of WDVV-like equations: the whole set (5).

**3.3** From (4) it is clear that $a_i$'s are defined modulo linear transformations (one can change $A$-cycle for any linear combination of them). Eqs.(5) possess adequate "covariance": the least trivial part is that $F_k$ can be substituted by $F_k + \sum_l \epsilon_l F_l$. Then

$$F_k^{-1} \to (F_k + \sum \epsilon_l F_l)^{-1} = F_k^{-1} - \sum \epsilon_l F_k^{-1} F_l F_k^{-1} + \sum \epsilon_l \epsilon_{l'} F_k^{-1} F_l F_k^{-1} F_{l'} F_k^{-1} + \dots$$

Clearly, (5) – valid for all triples of indices *simultaneously* – is enough to guarantee that $F_i(F_k + \sum \epsilon_l F_l)^{-1} F_j = F_j(F_k + \sum \epsilon_l F_l)^{-1} F_i$. Covariance under any replacement of $A$ and $B$-cycles together will be seen from the general proof in s.4 below: in fact the role of $F_k$ can be played by $F_{d\omega}$, associated with *any* holomorphic 1-differential $d\omega$ on the Riemann surface.

**3.4** For metric $\eta$, which is a second derivative,

$$\eta_{ij} = \frac{\partial^2 h}{\partial a_i \partial a_j} \equiv h_{,ij} \tag{8}$$

(as is the case for our $\eta^{(k)}_{mn} \equiv (F_k)_{mn}$: $h = h^{(k)} = \partial F / \partial a_k$), $\Gamma^i_{jk} = \frac{1}{2}\eta^{im} h_{,jkm}$ and the Riemann tensor

$$\begin{aligned} R^i_{jkl} &= \Gamma^i_{jl,k} + \Gamma^i_{kn}\Gamma^n_{jl} - (k \leftrightarrow l) = \frac{1}{2}\eta^{im} h_{,jklm} - \frac{1}{4}\eta^{ip} h_{,pnk}\eta^{nm} h_{,mjl} - (k \leftrightarrow l) = \\ &= -\Gamma^i_{kn}\Gamma^n_{jl} + (k \leftrightarrow l) = -\frac{1}{4}\eta^{ip} h_{,pnk}\eta^{nm} h_{,mjl} + (k \leftrightarrow l) \end{aligned} \tag{9}$$

In terms of the matrix $\eta = \{(\eta)_{kl}\}$ the zero-curvature condition $R_{ijkl} = 0$ would be

$$\eta_{,i}\eta^{-1}\eta_{,j} \overset{?}{=} \eta_{,j}\eta^{-1}\eta_{,i}. \tag{10}$$

This equation is remarkably similar to (5) and (7), but when $\eta^{(k)}_{ij} = F_{ijk}$ is substituted into (10), it contains the *fourth* derivatives of $F$:

$$F_{k,i} F_k^{-1} F_{k,j} \overset{?}{=} F_{k,j} F_k^{-1} F_{i,k} \qquad \forall i, j, k = 1, \dots, N-1 \tag{11}$$

(no summation over $k$ in this formula!), while (5) is expressed through the third derivatives only.

In ordinary topological theories $\eta^{(0)}$ is always flat, i.e. (11) holds for $k = 0$ along with (7) - and this allows one to choose "flat coordinates" where $\eta^{(0)} = const$. Sometimes - see Appendix B for an interesting example - *all* the metrics $\eta^{(k)}$ are flat simultaneously. However, explicit example of s.5.1.2 demonstrates that this is not always the case: in this example (quantum cohomologies of $CP^2$) eqs.(5) are true for all $k = 0, 1, 2$, but only $\eta^{(0)}$ is flat (satisfies (10)), while $\eta^{(1)}$ and $\eta^{(2)}$ lead to non-vanishing curvatures.

**3.5** Throughout this paper we do not include $\Lambda_{QCD}$ (the remnant of the dilaton v.e.v.) in the set of moduli. Thus, our prepotential is a function of $a_i$ alone and does not need to be a homogeneous function.

**3.6** It is well known that the conventional WDVV equations (7) are pretty restrictive: this is an overdetermined system of equations for a single function $F(t_i)$, and it is a kind of surprise that they possess any solutions at all, and in fact there exist vast variety of them (associated with Whitham hierarchies, topological models and quantum cohomologies). The set (5) is even more overdetermined than (7), since $k$ can take *any* value. Thus, it is even more surprising that the solutions still exist (in order to convince the reader, we supplement the formal proof in s.4 below by explicit examples in Appendices A and B).

Of course, (5) is tautologically true for $N = 2$ and $N = 3$, it becomes a non-trivial system for $N \geq 4$.

**3.7** Our proof in s.4 actually suggests that in majority of cases when the *ordinary* WDVV (7) is true, the whole system (5) holds automatically. This implies that this entire system should possess some interpretation in the spirit of hierarchies or hidden symmetries. It still remains to be found. The geometrical or cohomological origin of relations (5) also remains obscure.

**3.8** In this paper we discuss solutions to (5), provided by conventional topological theories and – as a far less trivial example – by the simplest Seiberg-Witten prepotentials.

We beleive that more solutions to (5) can arise from more sophisticated examples of the Seiberg-Witten theory ($\mathcal{N} = 2$ SUSY Yang-Mills with other groups and with matter supermultiplets); the most interesting should be the UV-finite models, when hyperelliptic surfaces (the double coverings of $CP^1$) are substituted by coverings of elliptic curve (torus), and a new elliptic parameter $\tau$ emerges.

If this conjecture is true, one can look for some relation between (5) and Picard-Fuchs equations, and then address to the issue of the WDVV equations for the prepotential, associated with families of the Calabi-Yau manifolds.

**3.9** Effective theory (1) is naively *non-topological*. From the 4-dimensional point of view it describes the low-energy limit of the Yang-Mills theory which – at least, in the $\mathcal{N} = 2$ supersymmetric case – is *not* topological and contains propagating massless particles. Still this theory is entirely defined by a prepotential, which – as we now see – possesses *all* essential properties of the prepotentials in topological theory. Thus, from the "stringy" point of view (when everything is described in terms of universality classes of effective actions) the Seiberg-Witten models belong to the same class as topological models: only the way to extract physically meaningful correlators from the prepotential is different. This can serve as a new evidence that the notion of topological theories is deeper than it is usually assumed: as emphasized in [4] it can be actually more related to the low-energy (IR) limit of field theories than to the property of the correlation functions to be constants in physical space-time.

**3.10** The issue of the WDVV equations in context of the Seiberg-Witten theory has been addressed in [7]. Unfortunately, we do not understand the statements in this paper and their relation to eqs.(5).

# 4 The proof of eqs.(5)

**4.1** Let us begin with reminding the proof of the WDVV equations (7) for ordinary topological theories. We take the simplest of all possible examples, when $\phi_i$ are polynomials of a single variable $\lambda$. The proof is essentially the check of consistency between the following formulas:

$$\phi_i(\lambda)\phi_j(\lambda) = C_{ij}^k \phi_k(\lambda) \bmod W'(\lambda), \tag{12}$$

$$F_{ijk} = \mathrm{res}\frac{\phi_i\phi_j\phi_k(\lambda)}{W'(\lambda)} = \sum_\alpha \frac{\phi_i\phi_j\phi_k(\lambda_\alpha)}{W''(\lambda_\alpha)}, \tag{13}$$

$$\eta_{kl} = \mathrm{res}\frac{\phi_k\phi_l(\lambda)}{W'(\lambda)} = \sum_\alpha \frac{\phi_k\phi_l(\lambda_\alpha)}{W''(\lambda_\alpha)}, \tag{14}$$

$$F_{ijk} = \eta_{kl}C_{ij}^l. \tag{15}$$

Here $\lambda_\alpha$ are the roots of $W'(\lambda)$.

In addition to the consistency of (12)-(15), one should know that *such* $F_{ijk}$, given by (13), are the third derivatives of a single function $F(a)$, i.e.

$$F_{ijk} = \frac{\partial^3 F}{\partial a_i \partial a_j \partial a_k}. \tag{16}$$

This integrability property of (13) follows from separate arguments and can be checked independently. But if (12)-(14) is given, the proof of (15) is straightforward:

$$\begin{aligned} \eta_{kl}C_{ij}^l &= \sum_\alpha \frac{\phi_k\phi_l(\lambda_\alpha)}{W''(\lambda_\alpha)} C_{ij}^l \overset{(12)}{=} \\ &= \sum_\alpha \frac{\phi_k(\lambda_\alpha)}{W''(\lambda_\alpha)} \phi_i(\lambda_\alpha)\phi_j(\lambda_\alpha) = F_{ijk}. \end{aligned} \tag{17}$$

Note that (12) is defined modulo $W'(\lambda)$, but $W'(\lambda_\alpha) = 0$ at all the points $\lambda_\alpha$.

Imagine now that we change the definition of the metric:

$$\eta_{kl} \to \eta_{kl}(\omega) = \sum_\alpha \frac{\phi_k\phi_l(\lambda_\alpha)}{W''(\lambda_\alpha)}\omega(\lambda_\alpha). \tag{18}$$

Then the WDVV equations would still be correct, provided the definition (12) of the algebra is also changed for

$$\phi_i(\lambda)\phi_j(\lambda) = C_{ij}^k(\omega)\phi_k(\lambda)\omega(\lambda) \bmod W'(\lambda). \tag{19}$$

This describes an associative algebra, whenever the polynomials $\omega(\lambda)$ and $W'(\lambda)$ are co-prime, i.e. do not have common divisors. Note that (13) – and thus the fact that $F_{ijk}$ is the third derivative of the same $F$ – remains intact! One can now take for $\omega(\lambda)$ any of the operators $\phi_k(\lambda)$, thus reproducing eqs.(5) for all topological theories [1] (see Appendix A for explicit example).

[1] To make (5) sensible, one should require that $W'(\lambda)$ has only *simple* zeroes, otherwise some of the matrices $F_k$ can be degenerate and non-invertible.

**4.2** In the case of the Seiberg-Witten model the polynomials $\phi_i(\lambda)$ are substituted by the canonical holomorphic differentials $d\omega_i(\lambda)$ on hyperelliptic surface (2). This surface can be represented in a standard hyperelliptic form,

$$y^2 = P_N^2(\lambda) - 1, \tag{20}$$

(where $y = \frac{1}{2}\left(w - \frac{1}{w}\right)$) and is of genus $g = N - 1$.[2]

**4.2.1** Instead of (12) and (19) we now put

$$d\omega_i(\lambda)d\omega_j(\lambda) = C_{ij}^k(d\omega)d\omega_k(\lambda)d\omega(\lambda) \bmod \frac{dP_N(\lambda)d\lambda}{y^2}. \tag{21}$$

In contrast to (19) we can not now choose $\omega = 1$ (to reproduce (12)), because now we need it to be a 1-differential. Instead we just take $d\omega$ to be a *holomorphic* 1-differential. However, there is no distinguished one – just a $g$-parametric family – and $d\omega$ can be *any* one from this family. We require only that it is co-prime with $\frac{dP_N(\lambda)}{y}$.

If the algebra (21) exists, the structure constants $C_{ij}^k(d\omega)$ satisfy the associativity condition (if $d\omega$ and $\frac{dP_N}{y}$ are co-prime). But we still need to show that it indeed exists, i.e. that if $d\omega$ is given, one can find ($\lambda$-independent) $C_{ij}^k$. This is a simple exercise: all $d\omega_i$ are linear combinations of

$$\begin{gathered} dv_k(\lambda) = \frac{\lambda^{k-1}d\lambda}{y}, \quad k = 1, \ldots, g: \\ dv_k(\lambda) = \sigma_{ki}d\omega_i(\lambda), \quad d\omega_i = (\sigma^{-1})_{ik}dv_k, \quad \sigma_{ki} = \oint_{A_i} dv_k, \end{gathered} \tag{22}$$

also $d\omega(\lambda) = s_k dv_k(\lambda)$. Thus, (21) is in fact a relation between the polynomials:

$$\left(\sigma_{ii'}^{-1}\lambda^{i'-1}\right)\left(\sigma_{jj'}^{-1}\lambda^{j'-1}\right) = C_{ij}^k\left(\sigma_{kk'}^{-1}\lambda^{k'-1}\right)\left(s_l\lambda^{l-1}\right) + p_{ij}(\lambda)P_N'(\lambda). \tag{23}$$

At the l.h.s. we have a polynomial of degree $2(g-1)$. Since $P_N'(\lambda)$ is a polynomial of degree $N-1 = g$, this implies that $p_{ij}(\lambda)$ should be a polynomial of degree $2(g-1) - g = g - 2$. The identification of two polynomials of degree $2(g-1)$ impose a set of $2g-1$ equations for the coefficients. We have a freedom to adjust $C_{ij}^k$ and $p_{ij}(\lambda)$ (with $i, j$ fixed), i.e. $g + (g-1) = 2g-1$ free parameters: exactly what is necessary. The linear system of equations is non-degenerate for co-prime $d\omega$ and $dP_N/y$.

Thus, we proved that the algebra (21) exists (for a given $d\omega$) – and thus $C_{ij}^k(d\omega)$ satisfy the associativity condition

$$C_i(d\omega)C_j(d\omega) = C_j(d\omega)C_i(d\omega). \tag{24}$$

**4.2.2** Instead of (13) we have [5]:

$$\begin{gathered} F_{ijk} = \frac{\partial^3 F}{\partial a_i \partial a_j \partial a_k} = \frac{\partial T_{ij}}{\partial a_k} = \\ = \operatorname*{res}_{d\lambda=0} \frac{d\omega_i d\omega_j d\omega_k}{d\lambda\left(\frac{dw}{w}\right)} = \operatorname*{res}_{d\lambda=0} \frac{d\omega_i d\omega_j d\omega_k}{d\lambda \frac{dP_N}{y}} = \sum_\alpha \frac{\hat{\omega}_i(\lambda_\alpha)\hat{\omega}_j(\lambda_\alpha)\hat{\omega}_k(\lambda_\alpha)}{P_N'(\lambda_\alpha)/\hat{y}(\lambda_\alpha)} \end{gathered} \tag{25}$$

[2] Note that in this way one defines a peculiar $g$-parametric family of hyperelliptic surfaces (the moduli space of *all* the Riemann surfaces has dimension $3g-3$, while that of all the hyperelliptic ones – $2g-1$). One can take for the $g$ moduli the set $\{h_k\}$ or instead the set of periods $\{a_i\}$. This particular family is associated with the Toda-chain hierarchy, $N$ being the length of the chain (while *all* the Riemann surfaces of all genera are associated with KP, and *all* the hyperelliptic ones – with KdV hierarchy).

The sum at the r.h.s. goes over all the $2g+2$ ramification points $\lambda_\alpha$ of the hyperelliptic curve (i.e. over the zeroes of $y^2 = P_N^2(\lambda) - 1 = \prod_{\alpha=1}^N (\lambda - \lambda_\alpha)$); $d\omega_i(\lambda) = (\hat{\omega}_i(\lambda_\alpha) + O(\lambda - \lambda_\alpha)) \frac{d\lambda}{\sqrt{\lambda - \lambda_\alpha}}$, $\quad \hat{y}^2(\lambda_\alpha) = \prod_{\beta \neq \alpha} (\lambda_\alpha - \lambda_\beta)$.

Though eq.(25) can be extracted from [5], for the sake of completeness we present a proof of this formula in Appendix C at the end of this paper.

**4.2.3** We define the metric in the following way:

$$\begin{aligned} \eta_{kl}(d\omega) &= \operatorname*{res}_{d\lambda=0} \frac{d\omega_k d\omega_l d\omega}{d\lambda \left(\frac{dw}{w}\right)} = \operatorname*{res}_{d\lambda=0} \frac{d\omega_k d\omega_l d\omega_k}{d\lambda \frac{dP_N}{y}} = \\ &= \sum_\alpha \frac{\hat{\omega}_k(\lambda_\alpha) \hat{\omega}_l(\lambda_\alpha) \hat{\omega}(\lambda_\alpha)}{P_N'(\lambda_\alpha)/\hat{y}(\lambda_\alpha)} \end{aligned} \tag{26}$$

In particular, for $d\omega = d\omega_k$, $\eta_{ij}(d\omega_k) = F_{ijk}$: this choice will give rise to (5).

Given (21), (25) and (26), one can check:

$$F_{ijk} = \eta_{kl}(d\omega) C_{ij}^k(d\omega). \tag{27}$$

Note that $F_{ijk} = \frac{\partial^3 F}{\partial a_i \partial a_j \partial a_k}$ at the l.h.s. of (27) is independent of $d\omega$! The r.h.s. of (27) is equal to:

$$\begin{aligned} \eta_{kl}(d\omega) C_{ij}^k(d\omega) &= \operatorname*{res}_{d\lambda=0} \frac{d\omega_k d\omega_l d\omega}{d\lambda \left(\frac{dw}{w}\right)} C_{ij}^l(d\omega) \stackrel{(21)}{=} \\ = \operatorname*{res}_{d\lambda=0} \frac{d\omega_k}{d\lambda \left(\frac{dw}{w}\right)} \left( d\omega_i d\omega_j - p_{ij} \frac{dP_N d\lambda}{y^2} \right) &= F_{ijk} - \operatorname*{res}_{d\lambda=0} \frac{d\omega_k}{d\lambda \left(\frac{dP_N}{y}\right)} p_{ij}(\lambda) \frac{dP_N d\lambda}{y^2} = \\ &= F_{ijk} - \operatorname*{res}_{d\lambda=0} \frac{p_{ij}(\lambda) d\omega_k(\lambda)}{y} \end{aligned} \tag{28}$$

It remains to prove that the last item is indeed vanishing for any $i, j, k$. This follows from the fact that $\frac{p_{ij}(\lambda) d\omega_k(\lambda)}{y}$ is singular only at zeroes of $y$, it is not singular at $\lambda = \infty$ because $p_{ij}(\lambda)$ is a polynomial of low enough degree $g - 2 < g + 1$. Thus the sum of its residues at ramification points is thus the sum over *all* the residues and therefore vanishes.

This completes the proof of associativity condition for any $d\omega$. Taking $d\omega = d\omega_k$ (which is obviously co-prime with $\frac{dP_N}{y}$), we obtain (5).

# 5 Appendix A. Explicit example of (5) for topological theory

In this appendix we address to the questions about the system (5) with the two goals: First, we provide explicit examples to convince the reader that entire system (5) is generically true for topological theories, not only (7), as one usually believes. Second – since one gets convinced – we ask if (5) is just a direct corollary of (7), supplemented by peculiar symmetry properties $(F_i)_{jk} = (F_j)_{ik} = (F_k)_{ij}$. We demonstrate that this is indeed the case for $g = N - 1 = 3$ (eqs.(5) are tautologically correct for $g = 1$ and $g = 2$). However, this does not seem to be the case for $g \geq 4$: (5) relies heavily on the fact that $F_{ijk}$ are the third derivatives, $F_{ijk} = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k}$, namely on relations like (13).

## 5.1 Examples for $g = N - 1 = 3$

**5.1.1** Let us begin with the topological model with $W'(\lambda) = \lambda^3 - q$ ($q \neq 0$ – the roots of $W'(\lambda)$ are all different – in order to avoid degeneracies of the matrices $F_1$ and $F_2$). In the basis $\phi_i = \lambda^i$, $i = 0, 1, 2$ one easily obtains from (13):

$$F_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & q \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & q \\ 0 & q & 0 \end{pmatrix}. \tag{29}$$

The corresponding prepotential is

$$F = \frac{1}{2}t_0 t_1^2 + \frac{1}{2}t_0^2 t_2 + \frac{q}{2}t_1 t_2^2. \tag{30}$$

The inverse matrices are $F_i^{-1}(q) = F_i(1/q)$.

In order to shorten the calculations it is useful to note that – since the matrices $F_i$ are symmetric – the relations (5) mean that all the matrices

$$U_{ikj} = F_i F_k^{-1} F_j : \qquad U_{ikj} = U_{ikj}^{tr}. \tag{31}$$

are also symmetric. It is a trivial exercise to check that

$$U_{102} = F_1 F_0^{-1} F_2 = qI, \quad U_{201} = qF_1, \quad U_{012} = F_1(q), \quad U_{210} = F_1(1/q),$$
$$U_{021} = U_{120} = \begin{pmatrix} 1/q & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{32}$$

are indeed all symmetric.

**5.1.2** Consider now a generalization of the previous example: the quantum cohomology of $CP^2$ [8]. The prepotential is

$$F = \frac{1}{2}t_0 t_1^2 + \frac{1}{2}t_0^2 t_2 + \sum_{n=1}^{\infty} \frac{N_n t_2^{3n-1}}{(3n-1)!} e^{nt_1} \tag{33}$$

and the corresponding matrices are:

$$F_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & F_{111} & F_{112} \\ 0 & F_{112} & F_{122} \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & F_{112} & F_{122} \\ 0 & F_{122} & F_{222} \end{pmatrix} \tag{34}$$

where

$$\begin{aligned} F_{111} &= \sum_n \frac{n^3 N_n}{(3n-1)!} t_2^{3n-1} e^{nt_1}, \\ F_{112} &= \sum_n \frac{n^2 N_n}{(3n-2)!} t_2^{3n-2} e^{nt_1}, \\ F_{122} &= \sum_n \frac{n N_n}{(3n-3)!} t_2^{3n-3} e^{nt_1}, \\ F_{222} &= \sum_n \frac{N_n}{(3n-4)!} t_2^{3n-4} e^{nt_1}. \end{aligned} \tag{35}$$

One can easily check that every equation in (5) is true if and only if

$$F_{222} = F_{112}^2 - F_{111}F_{122}. \tag{36}$$

Indeed,

$$F_1F_0^{-1}F_2 = \begin{pmatrix} 0 & F_{112} & F_{122} \\ F_{112} & F_{122}+F_{111}F_{112} & F_{222}+F_{111}F_{122} \\ F_{122} & F_{112}^2 & F_{112}F_{122} \end{pmatrix},$$

$$F_0F_1^{-1}F_2 = \frac{1}{F_{122}}\begin{pmatrix} -F_{112} & F_{122} & F_{222} \\ F_{122} & 0 & 0 \\ F_{112}^2 - F_{111}F_{122} & 0 & F_{122}^2 - F_{112}F_{222} \end{pmatrix}, \tag{37}$$

$$F_0F_2^{-1}F_1 = \frac{1}{F_{112}F_{222}-F_{122}^2}\begin{pmatrix} -F_{122} & F_{112}^2 - F_{111}F_{122} & 0 \\ F_{222} & F_{111}F_{222}-F_{112}F_{122} & F_{112}F_{222}-F_{122}^2 \\ 0 & F_{112}F_{222}-F_{122}^2 & 0 \end{pmatrix}$$

Eq.(36) is the famous equation, providing the recursive relations for $N_n$ [8]:

$$\frac{N_n}{(3n-4)!} = \sum_{a+b=n} \frac{a^2b(3b-1)b(2a-b)}{(3a-1)!(3b-1)!}N_aN_b. \tag{38}$$

For example, $N_2 = N_1^2$, $N_3 = 12N_1N_2 = 12N_1^3$, ...

The zero curvature condition (11) is obviously satisfied for $\eta^0 = F_0$: $R_{ijkl}(\eta^{(0)}) = 0$, but it is not fulfilled already for $\eta^{(1)} = F_1$:

$$R_{1212}(\eta^{(1)}) \sim F_{1112}F_{1222} - F_{1122}^2 = -N_1^2e^{3t_1} + \ldots \neq 0. \tag{39}$$

**5.1.3** Two above examples illustrate that – if (7), i.e. relation for $k=0$, is established – the equations (5) for all other $k$ hold as well. This of course follows – for the topological systems – from our analysis in s.4.1, but in fact for $g = N-1 = 3$ this is just an *arithmetic* property: one should only take into account the fact that $F_{ijk}$ is symmetric in all three indices.

Namely, let us write down the only non-trivial matrix element in relation (7):

$$(F_{11i}F_{22j} - F_{12i}F_{12j})(F_0^{-1})^{ij} = 0, \tag{40}$$

$(F_0^{-1})^{ij} = (\det\ F_0)^{-1}\hat{F}_0^{ij}$, where the entries in $\hat{F}_0$ are quadratic combinations of $F_{klm}$. Substituting the explicit expression for $\hat{F}_0$, we get for (40) certain sophisticated expression (too long to be presented here) through the 4-th powers of $F_{klm}$.

Now, do the same for the other eqs. in (5), e.g. for $U_{012}$: the only non-trivial matrix element is

$$(F_{00i}F_{22j} - F_{02i}F_{02j})(F_1^{-1})^{ij} = (\det\ F_1)^{-1} \times (\text{quartic combination of } F_{klm}). \tag{41}$$

One can check that the quartic combinations are literally the same in (40) and (41) – and in all other $U_{ijk}$, i.e. if any one of the equations (5) is satisfied, the others follow arithmetically.

## 5.2 $g > 3$

Thus, we see that for $g = N - 1 = 3$ any solution to the original WDVV eq. (7) is just *literally* solution to the whole system (5).

We now argue that for $g \geq 4$ this is – though generically true – but not for such a simple reason. Then we provide an analogue of the example from s.5.1.1 for $g \geq 4$ – which is now a little less trivial illustration.

**5.2.1** Let us try to repeat the reasoning from s.5.1.3 for generic $g$. The matrix element

$$\begin{gathered}\left(F_i F_k^{-1} F_j - F_j F_k^{-1} F_i\right)_{mn} = (F_{imr}F_{jns} - F_{inr}F_{jms})(F_k^{-1})^{rs} = \\ = (\det\ F_k)^{-1} \epsilon^{rr_1\ldots r_{g-1}} \epsilon^{ss_1\ldots s_{g-1}} (F_{imr}F_{jns} - F_{inr}F_{jms}) F_{kr_1s_1} \ldots F_{kr_{g-1}s_{g-1}}.\end{gathered} \tag{42}$$

If $k = 0$, but $i, j, m, n \neq 0$, the r.h.s. of (42) contains exactly $g+1$ indices "0" ($g-1$ times $k = 0$ plus exactly one of all the $r$'s and exactly one of all the $s$'s). Of indices $i, j, m, n$ at most two can be equal to 0 without making (42) vanishing identically. Thus, every item at the r.h.s. of (42) for $k = 0$ contains $g+1$, $g+2$ or $g+3$ indices "0".

If $k \neq 0$, and $i, j, m, n \neq 0$, the number of indices "0" at the r.h.s. is exactly 2 (one of all the $r$'s and one of all the $s$'s). Adding at most 2 indices "0" from among $i, j, m, n$ we get 2,3 or 4 such indices in every item if $k \neq 0$.

If entire system (5) with all $k$'s was *arithmetic* corollary of its subset (7) with $k = 0$ – as is the case for $g = 3$ in s.5.1.3 – the number of all indices, including "0", should match, i.e. $g+1$, $g+2$ or $g+3$ should coincide with 2,3 or 4. This restricts $g$ to be $g \leq 3$. For $g \geq 4$ the implication (7) $\Longrightarrow$ (5) – still true according to our consideration in s.4 – should be of more transcendental nature.

**5.2.2** Now we take the topological theory with $W'(\lambda) = \lambda^g - q$. In the basis $\phi_i = \lambda^i$, $i = 0, \ldots, g-1$ matrices $F_i$ are $g \times g$ analogs of (29), now units stand at the $i$-th upper skew-subdiagonal and $q$'s – at the $(q+1-i)$-th lower one so, again, $F_i^{-1}(q) = F_i(1/q)$: this is enough for explicit calculation.

For example, for $g = 4$ not only conventional combinations $U_{i0j} = F_i F_0^{-1} F_j$ are symmetric (i.e. satisfy (7), e.g.

$$U_{102} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 0 & q & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 0 & q & 0 \\ 0 & q & 0 & 0 \end{pmatrix}, \tag{43}$$

but the same is true, say, for

$$U_{123} = F_1 F_2^{-1} F_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/q \\ 0 & 0 & 1/q & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 0 & q & 0 \\ 0 & q & 0 & 0 \end{pmatrix} = \\ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 0 & q & 0 \end{pmatrix} \tag{44}$$

and all other $U_{ijk}$.

# 6 Appendix B. Explicit example of (5) related to the Seiberg-Witten effective theory

This example involves the leading (perturbative) approximation to the exact Seiberg-Witten prepotential, which – being the leading contribution – satisfies (5) by itself. The perturbative contribution is non-transcendental, thus calculation can be performed in explicit form:

$$F_{pert} \equiv F(a_i) = \frac{1}{2} \sum_{\substack{m<n \\ m,n=1}}^{N} (A_m - A_n)^2 \log(A_m - A_n) \Bigg|_{\sum_m A_m = 0} = \\ = \frac{1}{2} \sum_{\substack{i<j \\ i,j=1}}^{N-1} (a_i - a_j)^2 \log(a_i - a_j) + \frac{1}{2} \sum_{i=1}^{N-1} a_i^2 \log a_i \tag{45}$$

Here we took $a_i = A_i - A_N$ – one of the many possible choices of independent variables, which differ by linear transformations. According to comment **3.3** above, the system (5) is covariant under such changes.

We shall use the notation $a_{ij} = a_i - a_j$. The matrix

$$\{(F_1)_{mn}\} = \left\{ \frac{\partial^3 F}{\partial a_1 \partial a_m \partial a_n} \right\} = \\ = \begin{pmatrix} \frac{1}{a_1} + \sum\limits_{l\neq 1} \frac{1}{a_{1l}} & -\frac{1}{a_{12}} & -\frac{1}{a_{13}} & -\frac{1}{a_{14}} & \\ -\frac{1}{a_{12}} & +\frac{1}{a_{12}} & 0 & 0 & \\ -\frac{1}{a_{13}} & 0 & +\frac{1}{a_{13}} & 0 & \ldots \\ -\frac{1}{a_{14}} & 0 & 0 & +\frac{1}{a_{14}} & \\ & & \ldots & & \end{pmatrix} \tag{46}$$

i.e.,

$$\{(F_i)_{mn}\} = \frac{\delta_{mn}(1-\delta_{mi})(1-\delta_{ni})}{a_{im}} - \frac{\delta_{mi}(1-\delta_{ni})}{a_{in}} - \frac{\delta_{ni}(1-\delta_{mi})}{a_{im}} + \\ + \left(\frac{1}{a_i} + \sum_{l\neq i}\frac{1}{a_{ik}}\right)\delta_{mi}\delta_{ni} \tag{47}$$

The inverse matrix

$$\{(F_k^{-1})_{mn}\} = a_k + \delta_{mn}a_{km}(1-\delta_{mk}), \tag{48}$$

for example

$$\{(F_1^{-1})_{mn}\} = a_1 \begin{pmatrix} 1 & 1 & 1 & . \\ 1 & 1 & 1 & . \\ 1 & 1 & 1 & . \\ & \dots & & \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & . \\ 0 & a_{12} & 0 & . \\ 0 & 0 & a_{13} & . \\ & \dots & & \end{pmatrix} \tag{49}$$

As the simplest example let us consider the case $N = 4$. We already know from s.5.1.3 that for $N = 4$ it is sufficient to check only one of the eqs.(5), all the others follow automatically. We take $k = 1$. Then,

$$F_1 = \begin{pmatrix} \frac{1}{a_1}+\frac{1}{a_{12}}+\frac{1}{a_{13}} & -\frac{1}{a_{12}} & -\frac{1}{a_{13}} \\ -\frac{1}{a_{12}} & \frac{1}{a_{12}} & 0 \\ -\frac{1}{a_{13}} & 0 & \frac{1}{a_{13}} \end{pmatrix}, \quad F_2^{-1} = \begin{pmatrix} a_2+a_{21} & a_2 & a_2 \\ a_2 & a_2 & a_2 \\ a_2 & a_2 & a_2+a_{23} \end{pmatrix}, \\ F_3 = \begin{pmatrix} \frac{1}{a_{31}} & 0 & -\frac{1}{a_{31}} \\ 0 & \frac{1}{a_{32}} & -\frac{1}{a_{32}} \\ -\frac{1}{a_{31}} & -\frac{1}{a_{32}} & \frac{1}{a_3}+\frac{1}{a_{31}}+\frac{1}{a_{32}} \end{pmatrix} \tag{50}$$

and, say,

$$F_1F_2^{-1}F_3 = \begin{pmatrix} \star & -\frac{1}{a_{31}} & \Delta + \frac{a_{21}+a_{23}}{a_{13}^2} \\ -\frac{1}{a_{13}} & \star & \frac{1}{a_{13}} \\ \frac{a_{21}+a_{23}}{a_{13}^2} & \frac{1}{a_{13}} & \star \end{pmatrix} \tag{51}$$

where we do not write down manifestly the diagonal terms since, to check (5), one only needs to prove the symmetricity of the matrix. This is really the case, since

$$\Delta \equiv \frac{a_2}{a_1a_3} - \frac{a_{21}}{a_1a_{31}} - \frac{a_{23}}{a_3a_{13}} = 0 \tag{52}$$

Only at this stage we use manifestly that $a_{ij} = a_i - a_j$.

Now let us prove (5) for the general case. We check the equation for the inverse matrices. Namely, using

formulas (47)-(48), one obtains

$$
\begin{aligned}
(F_i^{-1}F_jF_k^{-1})_{\alpha\beta} = \frac{a_ia_k}{a_j} + \delta_{\alpha\beta}(1-\delta_{i\alpha})(1-\delta_{k\alpha})(1-\delta_{j\alpha})\frac{a_{i\alpha}a_{k\beta}}{a_{j\beta}} + \delta_{j\alpha}\delta_{j\beta}(1-\delta_{i\alpha})(1-\delta_{k\beta})\left(\frac{1}{a_j}+\sum_{n\neq j}\frac{1}{a_{jn}}\right)+ \\
+\delta_{j\alpha}(1-\delta_{i\alpha})a_{i\alpha}\left(\frac{a_k}{a_j}-\frac{a_{k\beta}}{a_{j\beta}}(1-\delta_{k\beta})(1-\delta_{j\beta})\right)+\delta_{j\beta}(1-\delta_{k\beta})\left(\frac{a_i}{a_j}-\frac{a_{i\alpha}}{a_{j\alpha}}(1-\delta_{i\alpha})(1-\delta_{j\alpha})\right) = \\
= \frac{a_ia_k}{a_j}+\delta_{\alpha\beta}(1-\delta_{i\alpha}-\delta_{k\alpha}-\delta_{j\alpha})\frac{a_{i\alpha}a_{k\beta}}{a_{j\beta}}+\delta_{j\alpha}\delta_{j\beta}\left(\frac{1}{a_j}+\sum_{n\neq j}\frac{1}{a_{jn}}\right)+ \\
+\delta_{j\alpha}a_{i\alpha}\left(\frac{a_k}{a_j}-\frac{a_{k\beta}}{a_{j\beta}}(1-\delta_{k\beta}-\delta_{j\beta})\right)+\delta_{j\beta}\left(\frac{a_i}{a_j}-\frac{a_{i\alpha}}{a_{j\alpha}}(1-\delta_{i\alpha}-\delta_{j\alpha})\right)
\end{aligned}
\tag{53}
$$

where we used that $i \neq j \neq k$. The first three terms are evidently symmetric with respect to interchanging $\alpha \leftrightarrow \beta$. In order to prove the symmetricity of the last two terms, we need to use the identities $\frac{a_k}{a_j} - \frac{a_{k\beta}}{a_{j\beta}} = \frac{a_\beta a_{jk}}{a_j a_{j\beta}} \stackrel{k=\beta}{\to} \frac{a_k}{a_j}$, $\frac{a_i}{a_j} - \frac{a_{i\alpha}}{a_{j\alpha}} = \frac{a_\alpha a_{ji}}{a_j a_{j\alpha}} \stackrel{i=\alpha}{\to} \frac{a_i}{a_j}$. Then, one gets

$$
\text{the last line of (53)} = \delta_{j\alpha}(1-\delta_{j\beta})\frac{a_{ij}a_{jk}}{a_j}\frac{a_\beta}{a_{j\beta}} + \delta_{j\beta}(1-\delta_{j\alpha})\frac{a_{ij}a_{jk}}{a_j}\frac{a_\alpha}{a_{j\alpha}} + \delta_{j\alpha}\delta_{j\beta}\frac{a_ka_{i\alpha}+a_ia_{k\beta}}{a_j} \tag{54}
$$

It is interesting to note (see also comment **3.4**) that in the particular example (45), all the metrics $\eta^{(k)}$ are flat. Moreover, it is easy to find the explicit flat coordinates:

$$
\begin{aligned}
\eta^{(k)} = \eta^{(k)}_{ij}da^ida^j = F_{ijk}da^ida_j = da_ida_j\partial^2_{ij}(\partial_kF) = \\
= \frac{da_k^2}{a_k}+\sum_{l\neq k}\frac{da_{kl}^2}{a_{kl}} = 4\left((d\sqrt{a_k})^2+\sum_{l\neq k}(d\sqrt{a_{kl}})\right).
\end{aligned}
\tag{55}
$$

# 7 Appendix C. The proof of eq.(23)

The crucial property of the differential $dS$ is that its variation with respect to moduli is holomorphic 1-differential: $\delta dS \cong$ *holomorphic*, in fact $\frac{\partial dS}{\partial a_i} \cong d\omega_i$.

From (4) it follows now [2] that the second derivative of the prepotential is period matrix of the Riemann surface:

$$
\frac{\partial^2 F}{\partial a_i \partial a_j} = \oint_{B_i}\frac{\partial dS}{\partial a_j} = \oint_{B_i} d\omega_j = T_{ij}. \tag{56}
$$

Thus, the third derivative

$$
\frac{\partial^3 F}{\partial a_i\partial a_j\partial a_k} = \frac{\partial T_{ij}}{\partial a_k}. \tag{57}
$$

It is very easy to evaluate the derivative of the period matrix of hyperelliptic curve w.r.t. the variation of any

ramification point $\lambda_\alpha$ [9][3]:

$$\frac{\partial T_{ij}}{\partial\lambda_\alpha} = \hat{\omega}_i(\lambda_\alpha)\hat{\omega}_j(\lambda_\alpha). \tag{58}$$

However, for the family (2) all the $2g+2$ ramification points depend only on $g$ moduli, thus we should also know $\frac{\partial\lambda_\alpha}{\partial a_k}$. This is easy to evaluate in two steps:

$$\frac{\partial\lambda_\alpha}{\partial a_k} = \frac{\partial\lambda_\alpha}{\partial h_l}\frac{\partial h_l}{\partial a_k}. \tag{59}$$

First step: the derivative

$$\frac{\partial a_k}{\partial h_l} = \oint_{A_k}\frac{\partial dS}{\partial h_l} \tag{60}$$

can be found from the explicit expression for $dS = \lambda\frac{dw}{w}$:

$$\frac{\partial dS}{\partial h_l} = \text{exact form } - \frac{d\lambda}{w}\frac{\partial w}{\partial h_l} = \text{exact form } - \frac{\lambda^{l-1}d\lambda}{y} \tag{61}$$

(since $\frac{dw}{w} = \frac{dP_N}{y}$ and $\frac{\partial P_N}{\partial h_l} = \lambda^{l-1}$). Thus

$$\frac{\partial a_k}{\partial h_l} = -\oint_{A_k} dv_l \overset{(22)}{=} -\sigma_{lk}, \tag{62}$$

and

$$\frac{\partial h_l}{\partial a_k} = -\sigma^{-1}_{kl}. \tag{63}$$

Second step: in order to evaluate $\frac{\partial\lambda_\alpha}{\partial h_l}$, let us take $h_l$-derivative of $P_N(\lambda) = \prod_\beta(\lambda-\lambda_\beta)$ and then put $\lambda = \lambda_\alpha$. We get first

$$\lambda^{l-1} = -P_N(\lambda)\sum_\beta \frac{\partial\lambda_\beta}{\partial h_l}\frac{1}{\lambda-\lambda_\beta}, \tag{64}$$

and the only term in the sum at the r.h.s. which contributes when $\lambda = \lambda_\alpha$ and $P_N(\lambda_\alpha) = 0$ is that with $\beta = \alpha$. Applying the L'Hôpital rule, we obtain:

$$\lambda_\alpha^{l-1} = -\frac{\partial\lambda_\alpha}{\partial h_l}P'_N(\lambda_\alpha), \tag{65}$$

or

$$\frac{\partial\lambda_\alpha}{\partial a_k} = \frac{\lambda_\alpha^{l-1}}{P'_N(\lambda_\alpha)}\sigma^{-1}_{kl} \overset{(22)}{=} \frac{\hat{y}(\lambda_\alpha)}{P'_N(\lambda_\alpha)}\hat{\omega}_k(\lambda_\alpha). \tag{66}$$

Together with (57) and (58) this finally gives:

$$\begin{aligned}\frac{\partial^3 F}{\partial a_i\partial a_j\partial a_k} &= \frac{\partial T_{ij}}{\partial a_k} = \sum_\alpha \frac{\partial T_{ij}}{\partial\lambda_\alpha}\frac{\partial\lambda_\alpha}{\partial a_k} = \\ &= \sum_\alpha \hat{\omega}_i(\lambda_\alpha)\hat{\omega}_j(\lambda_\alpha)\frac{\hat{\omega}_k(\lambda_\alpha)}{P'_N(\lambda_\alpha)/\hat{y}(\lambda_\alpha)}\end{aligned} \tag{67}$$

as stated in (25).

[3] Indeed, from (22) and (56), one obtains

$$\frac{\partial T_{ij}}{\partial\lambda_\alpha} = \sigma^{-1}_{ik}\sigma^{-1}_{jm}\left(\oint_{B_l}\frac{\partial v_k}{\partial\lambda_\alpha}\oint_{A_l}v_m - \oint_{A_l}\frac{\partial v_k}{\partial\lambda_\alpha}\oint_{B_l}v_m\right)$$

Using the local representation $v_m = du_m$, one gets

$$0 = \int v_m \wedge \frac{\partial v_k}{\partial\lambda_\alpha} = \int d\left(u_m\frac{\partial v_k}{\partial\lambda_\alpha}\right) = \oint_{B_l}\frac{\partial v_k}{\partial\lambda_\alpha}\oint_{A_l}v_m - \oint_{A_l}\frac{\partial v_k}{\partial\lambda_\alpha}\oint_{B_l}v_m - \operatorname*{res}_{\lambda_\alpha}\left(u_m\frac{\partial v_k}{\partial\lambda_\alpha}\right)$$

Therefore,

$$\frac{\partial T_{ij}}{\partial\lambda_\alpha} = \sigma^{-1}_{ik}\hat{v}_k(\lambda_\alpha)\sigma^{-1}_{jm}\hat{v}_m(\lambda_\alpha) = \hat{\omega}_i(\lambda_\alpha)\hat{\omega}_j(\lambda_\alpha)$$

where we used the expansion in the vicinity of the point $\lambda_\alpha$: $u_m = 2\hat{v}_m(\lambda_\alpha)\sqrt{\lambda-\lambda_\alpha}+\ldots$, $\frac{\partial v_k}{\partial\lambda_\alpha} = \frac{\hat{v}_k(\lambda_\alpha)}{\lambda-\lambda_\alpha}+\ldots$.

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